

Physics 137B, Spring 2004  
 Solutions to HW #2

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Problem 1

(a) We have 2 electrons  $\therefore s_1 = \frac{1}{2} \text{ & } s_2 = \frac{1}{2}$

$$\text{As we saw in class, } \begin{matrix} \frac{1}{2} \\ s_1 \end{matrix} \otimes \begin{matrix} \frac{1}{2} \\ s_2 \end{matrix} = \begin{matrix} 1 \\ j_1 \end{matrix} \oplus \begin{matrix} 0 \\ j_2 \end{matrix}$$

Since we were told that the electrons are in an eigenstate of  $J^2$  with eigenvalue  $2\hbar^2$ , we know that  $j = 1$  since

$$J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$\therefore J^2 |1, m\rangle = \hbar^2 1(1+1) |1, m\rangle = 2\hbar^2 |1, m\rangle.$$

So the electrons are in one of the 3 states corresponding to  $j=1$

$|1, 1\rangle, |1, 0\rangle, |1, -1\rangle$ , but we'll just call it  $|1, m\rangle$ . (or a linear combination of them).

We are asked to determine the value of  $\vec{s}_1 \cdot \vec{s}_2$ . First notice that

$$\begin{aligned} J^2 &= (\vec{s}_1 + \vec{s}_2)^2 = (\vec{s}_1 + \vec{s}_2) \cdot (\vec{s}_1 + \vec{s}_2) = s_1^2 + \vec{s}_1 \cdot \vec{s}_2 + \vec{s}_2 \cdot \vec{s}_1 + s_2^2 \\ &= s_1^2 + s_2^2 + 2(\vec{s}_1 \cdot \vec{s}_2) \quad (\text{since } \vec{s}_1 \text{ & } \vec{s}_2 \text{ commute as the problem states}). \end{aligned}$$

$$\therefore \vec{s}_1 \cdot \vec{s}_2 = \frac{1}{2} [\vec{J}^2 - s_1^2 - s_2^2]$$

$$\vec{s}_1 \cdot \vec{s}_2 |1, m\rangle = \frac{1}{2} [\vec{J}^2 - s_1^2 - s_2^2] |1, m\rangle = \frac{1}{2} [J^2 |1, m\rangle - s_1^2 |1, m\rangle - s_2^2 |1, m\rangle]$$

$$\therefore J^2 |1, m\rangle = 2\hbar^2 |1, m\rangle$$

$$\therefore s_1^2 |1, m\rangle = s_1^2 |1, m\rangle = \hbar^2 \frac{1}{2} \left(\frac{1}{2}+1\right) |1, m\rangle = \frac{3}{4} \hbar^2 |1, m\rangle$$

$$\therefore s_2^2 |1, m\rangle = s_2^2 |1, m\rangle = \hbar^2 \frac{1}{2} \left(\frac{1}{2}+1\right) |1, m\rangle = \frac{3}{4} \hbar^2 |1, m\rangle$$

Let me explain the last two steps. Recall that  $|1, m\rangle$  can be written as  $\frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow)$  in the prof. notation, where the arrows correspond to  $m$  values.

But as far as  $S_1^2$  &  $S_2^2$  are concerned the  $m$  values do not matter; only the  $s$  values do. But since both particles are spin  $\frac{1}{2}$ ,  $S_1^2 \left\{ \begin{array}{c} \uparrow\downarrow \\ \downarrow\uparrow \end{array} \right. = \frac{3}{4}\hbar^2 \left\{ \begin{array}{c} \uparrow\uparrow \\ \downarrow\downarrow \end{array} \right.$   
Similarly for  $S_2^2$ .

$$\therefore (\vec{S}_1 \cdot \vec{S}_2) |1, m\rangle = \frac{1}{2} \left[ 2\hbar^2 + 2 \left( \frac{3}{4}\hbar^2 \right) \right] |1, m\rangle = \frac{1}{4} \hbar^2 |1, m\rangle$$

$$(b) \langle J_z \rangle = 0 \quad \therefore \text{Our state is } |1, 0\rangle = \frac{1}{\sqrt{2}} (|1\downarrow\rangle + |\downarrow 1\rangle)$$

$$\text{First of all, from part a } \langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \langle 1, m | \vec{S}_1 \cdot \vec{S}_2 | 1, m \rangle = \frac{1}{4} \hbar^2.$$

$$\text{Let's see if this agrees with } \langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \langle S_{1x} S_{2x} \rangle + \langle S_{1y} S_{2y} \rangle + \langle S_{1z} S_{2z} \rangle$$

$$\text{Before I compute all this, recall: } S_x | \frac{1}{2}, \frac{1}{2} \rangle = \frac{\hbar}{2} | \frac{1}{2}, -\frac{1}{2} \rangle \text{ (or } S_x \uparrow = \frac{\hbar}{2} \downarrow \text{)}$$

$$S_x | \frac{1}{2}, -\frac{1}{2} \rangle = \frac{\hbar}{2} | \frac{1}{2}, \frac{1}{2} \rangle$$

$$S_y | \frac{1}{2}, \frac{1}{2} \rangle = -\frac{\hbar}{2i} | \frac{1}{2}, -\frac{1}{2} \rangle$$

$$S_y | \frac{1}{2}, -\frac{1}{2} \rangle = \frac{\hbar}{2i} | \frac{1}{2}, \frac{1}{2} \rangle$$

$$S_z | \frac{1}{2}, \frac{1}{2} \rangle = \frac{\hbar}{2} | \frac{1}{2}, \frac{1}{2} \rangle$$

$$S_z | \frac{1}{2}, -\frac{1}{2} \rangle = -\frac{\hbar}{2} | \frac{1}{2}, -\frac{1}{2} \rangle$$

If you don't remember where these equations come from, you can derive them using (a) the matrix form of the operators  $| \frac{1}{2}, \frac{1}{2} \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $| \frac{1}{2}, -\frac{1}{2} \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  or  
(b) the raising and lowering operator ( $S_x = \frac{1}{2}(S_+ + S_-)$ ,  $S_y = \frac{1}{2i}(S_+ - S_-)$ ).

$$\cdot S_{1x} S_{2x} |1, 0\rangle = S_{1x} S_{2x} \frac{1}{\sqrt{2}} (|1\downarrow\rangle + |\downarrow 1\rangle) = \frac{1}{\sqrt{2}} S_{1x} \left[ \frac{\hbar}{2} |1\uparrow\rangle + \frac{\hbar}{2} |\downarrow 1\rangle \right]$$

$$= \frac{\hbar}{2\sqrt{2}} S_{1x} [|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] = \frac{\hbar}{2\sqrt{2}} \left[ \frac{\hbar}{2} |\downarrow\uparrow\rangle + \frac{\hbar}{2} |\uparrow\downarrow\rangle \right] = \frac{\hbar^2}{4\sqrt{2}} [|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] = \frac{\hbar^2}{4} |1, 0\rangle$$

$$\cdot S_{1y} S_{2y} |1, 0\rangle = S_{1y} S_{2y} \frac{1}{\sqrt{2}} (|1\downarrow\rangle + |\downarrow 1\rangle) = \frac{1}{\sqrt{2}} S_{1y} \left[ \frac{\hbar}{2i} |1\uparrow\rangle - \frac{\hbar}{2i} |\downarrow 1\rangle \right] = \frac{\hbar}{i2\sqrt{2}} [|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle]$$

$$= \frac{\hbar}{i2\sqrt{2}} \left[ -\frac{\hbar}{2i} |\downarrow\uparrow\rangle - \frac{\hbar}{2i} |\uparrow\downarrow\rangle \right] = \frac{-\hbar^2}{4i^2 4\sqrt{2}} [|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] = \frac{\hbar^2}{4\sqrt{2}} [|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] = \frac{\hbar^2}{4} |1, 0\rangle$$

$$\langle S_{1z} S_{2z} | 1,0 \rangle = S_{1z} S_{2z} \frac{1}{\sqrt{2}} (| \uparrow\downarrow \rangle + | \downarrow\uparrow \rangle) = \frac{\hbar}{2\sqrt{2}} S_{1z} (-|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{\hbar^2}{4\sqrt{2}} (-|\uparrow\downarrow\rangle - |\uparrow\uparrow\rangle)$$

$$= -\frac{\hbar^2}{4} \frac{1}{\sqrt{2}} (| \uparrow\downarrow \rangle + | \downarrow\uparrow \rangle) = -\frac{\hbar^2}{4} | 1,0 \rangle$$

$$\therefore \langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \langle S_{1x} S_{2x} \rangle + \langle S_{1y} S_{2y} \rangle + \langle S_{1z} S_{2z} \rangle = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = \frac{\hbar^2}{4} \quad \checkmark$$

## Problem 2

$$\hat{H} = -\gamma \vec{S} \cdot \vec{B}$$

$$\text{Recall for an operator } \hat{Q}: \frac{d}{dt} \langle \hat{Q} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle$$

$$[\hat{H}, \vec{S}] = -\gamma \left[ \vec{S} \cdot \vec{B}, \vec{S} \right] = -\gamma \begin{pmatrix} [S_x B_x + S_y B_y + S_z B_z, S_x] \\ [S_x B_x + S_y B_y + S_z B_z, S_y] \\ [S_x B_x + S_y B_y + S_z B_z, S_z] \end{pmatrix}$$

operator      vector      operator

$$= -\gamma \begin{pmatrix} B_y [S_y, S_x] + B_z [S_z, S_x] \\ B_x [S_x, S_y] + B_z [S_z, S_y] \\ B_x [S_x, S_z] + B_y [S_y, S_z] \end{pmatrix} = -\gamma \begin{pmatrix} -i\hbar B_y S_z + i\hbar B_z S_y \\ i\hbar B_x S_z - i\hbar B_z S_x \\ i\hbar B_x S_y + i\hbar B_y S_x \end{pmatrix}$$

$$= -\gamma i\hbar \begin{pmatrix} B_z S_y - B_y S_z \\ B_x S_z - B_z S_x \\ B_y S_x - B_x S_y \end{pmatrix} = -\gamma i\hbar \vec{S} \times \vec{B}$$

$$\therefore \frac{d}{dt} \langle \vec{S} \rangle = \frac{i}{\hbar} (-\gamma i\hbar) \langle \vec{S} \times \vec{B} \rangle = \gamma \langle \vec{S} \rangle \times \vec{B}$$

More elegant way: For those of you who have taken 105 & know a little about tensor analysis (I will use Einstein notation, ie.  $B_j S_j = \sum_j B_j S_j$ ):

$$-\gamma [\vec{S} \cdot \vec{B}, \vec{S}]_i = -\gamma [S_j B_j, S_i] = -\gamma B_j [S_j, S_i] = -i\hbar \gamma B_j \epsilon_{jik} S_k$$

$$= +i\hbar \gamma \epsilon_{ijk} B_j S_k = -i\hbar \gamma \epsilon_{ikj} S_k B_j = -i\hbar \gamma [\vec{S} \times \vec{B}]_i$$

### Problem 3

$j = \frac{3}{2}$ , we have 4 states :  $\left| \frac{3}{2}, \frac{3}{2} \right\rangle, \left| \frac{3}{2}, \frac{1}{2} \right\rangle, \left| \frac{3}{2}, -\frac{1}{2} \right\rangle, \left| \frac{3}{2}, -\frac{3}{2} \right\rangle$   
 that we will choose  
 as our basis vectors

$$\begin{array}{c} \downarrow \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \downarrow \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \downarrow \\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \downarrow \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

We know that:  $J_z \left| \frac{3}{2}, \frac{3}{2} \right\rangle = \frac{3}{2}\hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle, J_z \left| \frac{3}{2}, \frac{1}{2} \right\rangle = \frac{1}{2}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle$   
 $J_z \left| \frac{3}{2}, -\frac{1}{2} \right\rangle = -\frac{1}{2}\hbar \left| \frac{3}{2}, -\frac{1}{2} \right\rangle, J_z \left| \frac{3}{2}, -\frac{3}{2} \right\rangle = -\frac{3}{2}\hbar \left| \frac{3}{2}, -\frac{3}{2} \right\rangle$

Recall that for the matrix :  $(J_z)_{nm} = \langle n | J_z | m \rangle$  where  $n \neq m$  are the basis vectors.

Using this fact & the action of  $J_z$  on the basis vectors:

$$J_z = \hbar \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \quad (J_z \text{ is diagonal in this basis w/ eigenvalues } \pm \frac{3}{2}\hbar, \pm \frac{1}{2}\hbar)$$

why? Because I chose the eigenvectors of  $J_z$  as my basis to begin with).

$$\text{Since } J_{\pm} = J_x \pm i J_y \rightarrow J_x = \frac{1}{2}(J_+ + J_-); J_- = \frac{1}{2i}(J_+ - J_-)$$

So let's find the matrix representation of  $J_+$  &  $J_-$  first recalling the fact that  $J_{\pm} | j, m \rangle = \hbar \sqrt{j(j \mp 1) - m(m \mp 1)} | j, m \mp 1 \rangle$

$$\therefore J_+ \left| \frac{3}{2}, \frac{3}{2} \right\rangle = 0, J_+ \left| \frac{3}{2}, \frac{1}{2} \right\rangle = \hbar \sqrt{3} \left| \frac{3}{2}, \frac{3}{2} \right\rangle, J_+ \left| \frac{3}{2}, -\frac{1}{2} \right\rangle = 2\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle, J_+ \left| \frac{3}{2}, -\frac{3}{2} \right\rangle = \hbar \sqrt{3} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle$$

$$J_- \left| \frac{3}{2}, \frac{3}{2} \right\rangle = \hbar \sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right\rangle, J_- \left| \frac{3}{2}, \frac{1}{2} \right\rangle = 2\hbar \left| \frac{3}{2}, -\frac{1}{2} \right\rangle, J_- \left| \frac{3}{2}, -\frac{1}{2} \right\rangle = \hbar \sqrt{3} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle, J_- \left| \frac{3}{2}, -\frac{3}{2} \right\rangle = 0$$

$$\therefore J_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}; J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\therefore J_x = \frac{\hbar}{2} \left[ \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \right] = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{2i} \left[ \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \right] = \frac{\hbar}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

### Problem 4

$$S_1 = \frac{3}{2}, \quad S_2 = \frac{1}{2} \quad : \quad \frac{3}{2} \otimes \frac{1}{2} = 2 \oplus 1$$

$$\therefore S_{\text{tot}} = 2 \text{ or } 1$$

We start with  $S_{\text{tot}} = 2$  states:

$$\begin{aligned} |2, 2\rangle &= \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle \\ |2, -2\rangle &= \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{aligned} \quad \left. \right\} (\text{b/c } m_{\text{tot}} = m_{S_1} + m_{S_2})$$

$$S_{\text{tot}}^- |2, 2\rangle = 2\hbar |2, 1\rangle \Rightarrow |2, 1\rangle = \frac{1}{2\hbar} S_{\text{tot}}^- |2, 2\rangle = \frac{1}{2\hbar} (S_1^- + S_2^-) \left( \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle \right)$$

$$= \frac{1}{2\hbar} \left[ \hbar\sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right]$$

$$|2, 1\rangle = \frac{\sqrt{3}}{2} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \frac{1}{2} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$S_{\text{tot}}^- |2, 1\rangle = \sqrt{6}\hbar |2, 0\rangle \Rightarrow |2, 0\rangle = \frac{1}{\sqrt{6}\hbar} S_{\text{tot}}^- |2, 1\rangle = \frac{1}{\sqrt{6}\hbar} (S_1^- + S_2^-) \left[ \frac{\sqrt{3}}{2} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \frac{1}{2} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right]$$

$$= \frac{1}{\sqrt{6}\hbar} \left[ \frac{\sqrt{3}}{2} \cdot 2\hbar \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \frac{\sqrt{3}}{2} \hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{1}{2} \hbar\sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + 0 \right]$$

$$= \frac{1}{\sqrt{2}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \frac{1}{\sqrt{6}\hbar} \sqrt{3}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$\therefore |2, 0\rangle = \frac{1}{\sqrt{2}} \left[ \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right]$$

$$S_{\text{tot}}^+ |2, -2\rangle = 2\hbar |2, -1\rangle \Rightarrow |2, -1\rangle = \frac{1}{2\hbar} S_{\text{tot}}^+ |2, -2\rangle = \frac{1}{2\hbar} (S_1^+ + S_2^+) \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$= \frac{1}{2\hbar} \left[ \sqrt{3}\hbar \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \hbar \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle \right]$$

$$\therefore |2, -1\rangle = \frac{\sqrt{3}}{2} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \frac{1}{2} \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

Now that we have found all the  $s_{\text{TOT}}=2$  states, we must find the  $s_{\text{TOT}}=1$  states.

For  $|1,1\rangle$ ,  $m_{s_{\text{TOT}}} = m_{s_1} + m_{s_2}$   $\therefore$  the vectors that go into the linear combination are  $|\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle$  &  $|\frac{3}{2}, \frac{3}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$

Also, notice that  $\langle 1,1 | 2,1 \rangle = 0$

Given these facts: 
$$\boxed{\langle 1,1 | = \frac{\sqrt{3}}{2} |\frac{3}{2}, \frac{3}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{2} |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle}$$

$$\begin{aligned} S_{\text{TOT}}^- |1,1\rangle &= \hbar \sqrt{2} |1,0\rangle \Rightarrow |1,0\rangle = \frac{1}{\hbar \sqrt{2}} S_{\text{TOT}}^- |1,1\rangle = \frac{1}{\hbar \sqrt{2}} (S_1^- + S_2^-) \left[ \frac{\sqrt{3}}{2} |\frac{3}{2}, \frac{3}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{2} |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \right] \\ &= \frac{1}{\hbar \sqrt{2}} \left[ \frac{\sqrt{3}}{2} \hbar \sqrt{3} |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle + 0 - \frac{1}{2} 2\hbar |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle - \frac{1}{2} \hbar |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \right] \\ &= \boxed{\frac{1}{\sqrt{2}} \left[ |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \right] = |1,0\rangle} \end{aligned}$$

$$\begin{aligned} S_{\text{TOT}}^- |1,0\rangle &= \sqrt{2} \hbar |1,-1\rangle \Rightarrow |1,-1\rangle = \frac{1}{\sqrt{2} \hbar} (S_1^+ + S_2^+) \frac{1}{\sqrt{2}} \left[ |\frac{3}{2}, \frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \right] \\ &= \frac{1}{2 \hbar} \left[ 2\hbar |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle + 0 - \sqrt{3} \hbar |\frac{3}{2}, -\frac{3}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle - \hbar |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle \right] \\ &= \boxed{\frac{1}{2} \left[ |\frac{3}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - \frac{\sqrt{3}}{2} |\frac{3}{2}, -\frac{3}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle \right] = |1,-1\rangle} \end{aligned}$$

### Problem 5

(a) Particle A  $|1,0\rangle$

Particle B  $|1,-1\rangle$

$$|1,0\rangle |1,-1\rangle = \frac{1}{\sqrt{2}} \left[ |2,-1\rangle + |1,-1\rangle \right]$$

$$P_{j_{\text{tot}}=2} = \frac{1}{2}$$

(b)  $|s, m_s\rangle = |\frac{3}{2}, \frac{1}{2}\rangle ; |l, m_l\rangle = |1, 0\rangle$

$$|\frac{3}{2}, \frac{1}{2}\rangle |1,0\rangle = \sqrt{\frac{3}{5}} |\frac{5}{2}, \frac{1}{2}\rangle + \frac{1}{\sqrt{15}} |\frac{3}{2}, \frac{1}{2}\rangle - \sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}\rangle$$

## Problem 6

$$H = \underbrace{\frac{p^2}{2m}}_{H_0} + \frac{1}{2} kx^2 + bx^4$$

$\sim$

$$H'$$

$$E_n^{(0)} = \hbar\omega \left(n + \frac{1}{2}\right) \rightarrow E_3^{(0)} = \frac{7}{2}\hbar\omega$$

$$E_3^{(1)} = \langle 3 | H' | 3 \rangle = b \langle 3 | x^4 | 3 \rangle$$

$$\text{Recall } a = \frac{1}{\sqrt{2}} \left[ \left( \frac{m\omega}{\hbar} \right)^{1/2} \hat{x} + i \frac{\hat{p}}{(m\omega)^{1/2}} \right]$$

$$a^\dagger = \frac{1}{\sqrt{2}} \left[ \left( \frac{m\omega}{\hbar} \right)^{1/2} \hat{x} - i \frac{\hat{p}}{(m\omega)^{1/2}} \right]$$

$$\therefore \hat{x} = \left( \frac{\hbar}{2m\omega} \right)^{1/2} (a^\dagger + a) \quad , \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

$$\begin{aligned} \hat{x}^4 &= \left( \frac{\hbar}{2m\omega} \right)^2 (a^\dagger + a)^4 = \left( \frac{\hbar}{2m\omega} \right)^2 (a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2)(a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2) \\ &= \frac{\hbar^2}{4m^2\omega^2} \left( a^{\dagger 4} + a^{\dagger 3}a + a^{\dagger 2}aa^\dagger + a^{\dagger 2}a^2 + a a a a^\dagger + a^\dagger a a a + a^\dagger a^2 a^\dagger + a^\dagger a^3 \right. \\ &\quad \left. + a a^3 + a a^\dagger a + a a^\dagger a a^\dagger + a a a^2 + a^2 a^\dagger + a^2 a a^\dagger + a^3 a^\dagger + a^4 \right) \end{aligned}$$

Notice that if the number of a  $\hat{a}$  or  $a^\dagger$  in a given term is not equal then the matrix element will go to zero since you will get  $\langle n | m \rangle \underset{=0}{\approx}$

$$\begin{aligned} \langle 3 | x^4 | 3 \rangle &= \langle 3 | a^{\dagger 2}a^2 + (a^\dagger a)^2 + a^\dagger a^2 a^\dagger + a a^\dagger a^2 + (a a^\dagger)^2 + a^3 a^\dagger + a^2 a^\dagger | 3 \rangle \frac{\hbar^2}{4m^2\omega^2} \\ \cdot \langle 3 | a^{\dagger 2}a^2 | 3 \rangle &= \langle 3 | a^{\dagger 2} \sqrt{3}\sqrt{2} | 1 \rangle = \sqrt{6} \langle 3 | \sqrt{3}\sqrt{2} | 3 \rangle = 6 \langle 3 | 3 \rangle = 6 \\ \cdot \langle 3 | a^\dagger a a^\dagger | 3 \rangle &= \langle 3 | a^\dagger a \sqrt{3}\sqrt{3} | 3 \rangle = 3 \langle 3 | a^\dagger a | 3 \rangle = 9 \langle 3 | 3 \rangle = 9 \\ \cdot \langle 3 | a^\dagger a^2 a^\dagger | 3 \rangle &= \langle 3 | a^\dagger a \sqrt{4}\sqrt{4} | 3 \rangle = 4 \langle 3 | \sqrt{4}\sqrt{4} | 3 \rangle = 12 \langle 3 | 3 \rangle = 12 \\ \cdot \langle 3 | a a^\dagger a^2 | 3 \rangle &= \langle 3 | a a^\dagger \sqrt{3}\sqrt{3} | 3 \rangle = 3 \langle 3 | \sqrt{3}\sqrt{3} | 3 \rangle = 12 \end{aligned}$$

$$\begin{aligned} \cdot \langle 3 | a a^\dagger a a^\dagger | 3 \rangle &= \langle 3 | a a^\dagger \sqrt{4} \sqrt{4} | 3 \rangle = 4 \langle 3 | \sqrt{4} \sqrt{4} | 3 \rangle = 16 \\ \cdot \langle 3 | a^2 a^\dagger a^2 | 3 \rangle &= \langle 3 | a^2 \sqrt{4} \sqrt{5} | 5 \rangle = \sqrt{20} \langle 3 | \sqrt{5} \sqrt{4} | 3 \rangle = 20 \end{aligned}$$

$$\therefore b \langle 3 | x^4 | 3 \rangle = b \frac{\hbar^2}{4m^2\omega^2} (6 + 9 + 12 + 12 + 16 + 20) = \frac{75}{4} \frac{b\hbar^2}{m^2\omega^2}$$

$$\therefore E_3 = E_3^{(0)} + E_3^{(1)} = \frac{7}{2} \hbar\omega + \frac{75}{4} b \frac{\hbar^2}{m^2\omega^2}$$

Now, we need to find the correction to the wavefunction:

$$\Psi_3^{(1)} = \sum_{l \neq 3} \frac{\langle l | H' | 3 \rangle}{E_3^{(0)} - E_l^{(0)}} | l^{(0)} \rangle ; \quad E_3^{(0)} - E_l^{(0)} = \frac{7}{2} \hbar\omega - \hbar\omega (l + \frac{1}{2}) = \hbar\omega \left[ \frac{7}{2} - l - \frac{1}{2} \right] = \hbar\omega (3 - l)$$

$\cdot \langle l | H' | 3 \rangle = b \langle l | x^4 | 3 \rangle$ . There are 16 terms to consider. However, the terms that will require  $l=3$  are excluded from the sum. Notice that the terms that will force  $l=3$  are exactly the terms we used for the energy. So, we consider all the terms except those.

$$\begin{aligned} \cdot \langle l | a^{+4} | 3 \rangle &= \sqrt{4 \cdot 5 \cdot 6 \cdot 7} \langle l | 7 \rangle = \sqrt{840} \delta_{l,7} \left( \text{i.e. } = \begin{cases} \sqrt{840} & \text{if } l=7 \\ 0 & \text{if } l \neq 7 \end{cases} \right) \\ \cdot \langle l | a^{+3} a | 3 \rangle &= \sqrt{3 \cdot 4 \cdot 5} \langle l | 5 \rangle = 3\sqrt{20} \delta_{l,5} \\ \cdot \langle l | a^{+2} a a^\dagger | 3 \rangle &= \sqrt{4 \cdot 4 \cdot 4 \cdot 5} \langle l | 5 \rangle = 4\sqrt{20} \delta_{l,5} \\ \cdot \langle l | a a^\dagger a a^\dagger | 3 \rangle &= \sqrt{4 \cdot 5 \cdot 5 \cdot 5} \langle l | 5 \rangle = 5\sqrt{20} \delta_{l,5} \\ \cdot \langle l | a^\dagger a^3 | 3 \rangle &= \sqrt{3 \cdot 2 \cdot 1 \cdot 1} \langle l | 1 \rangle = \sqrt{6} \delta_{l,1} \\ \cdot \langle l | a a^\dagger a^3 | 3 \rangle &= \sqrt{4 \cdot 5 \cdot 6 \cdot 6} \langle l | 5 \rangle = 6\sqrt{20} \delta_{l,5} \\ \cdot \langle l | a a^\dagger a^2 | 3 \rangle &= \sqrt{3 \cdot 2 \cdot 2 \cdot 2} \langle l | 1 \rangle = 2\sqrt{6} \delta_{l,1} \\ \cdot \langle l | a^2 a^\dagger a | 3 \rangle &= \sqrt{3 \cdot 3 \cdot 3 \cdot 2} \langle l | 1 \rangle = 3\sqrt{6} \delta_{l,1} \\ \cdot \langle l | a^3 a^\dagger | 3 \rangle &= \sqrt{4 \cdot 4 \cdot 3 \cdot 2} \langle l | 1 \rangle = 4\sqrt{6} \delta_{l,1} \\ \cdot \langle l | a^4 | 3 \rangle &= 0 \end{aligned}$$

$$\cdot \langle \Psi_3^{(1)} \rangle = \sum_{l \neq 3} \frac{b \frac{\hbar^2}{m^2\omega^2}}{\hbar\omega (3-l)} \left[ \sqrt{840} \delta_{l,7} + 18\sqrt{20} \delta_{l,5} + 10\sqrt{6} \delta_{l,1} \right] | l^{(0)} \rangle$$

Recall that Kronecker delta's collapse sums to one term.

$$= \frac{bt}{4m^2\omega^3} \left[ \frac{\sqrt{840}}{-4} |7\rangle - \frac{18\sqrt{20}}{2} |5\rangle + \frac{10\sqrt{6}}{2} |1\rangle \right] = \frac{bt}{4m^2\omega^3} \left[ \frac{5\sqrt{6}}{1} |1\rangle - \frac{9\sqrt{20}}{1} |5\rangle - \sqrt{\frac{210}{4}} |1\rangle \right]$$

$$\therefore |\psi\rangle = |3\rangle + \frac{bt}{4m^2\omega^3} \left[ 5\sqrt{6} |1\rangle - 9\sqrt{20} |5\rangle - \sqrt{\frac{210}{4}} |1\rangle \right]$$